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# The proportion of various graphs in graph-designs

Richard M. Wilson

ABSTRACT. Let $\mathcal{G}$ be a family of simple graphs. A $\mathcal{G}$-design on $n$ points is a decomposition of the edges of $K_n$ into copies of graphs in $\mathcal{G}$. In case that $\mathcal{G}$ consists of complete graphs $K_k$ with $k$ in some set $K$ of positive integers, such a $\mathcal{G}$-design is called a pairwise balanced design (PBD) on $n$ points with block sizes from $K$. Here we are concerned with the possible proportions of the numbers of copies of graphs $G \in \mathcal{G}$ that appear in decompositions for large $n$. We extend a result of Colbourn and Rödl on PBDs to $\mathcal{G}$-designs, and give a further result on the possible numbers of copies of $G$ in a $\mathcal{G}$-design containing each vertex of the complete graph $K_n$.

## 1. Introduction

For a positive integer $n$ and a set $K$ of positive integers, a 2-$(n, K, 1)$ *design* consists of a set $X$ of $n$ points and a family $\mathcal{A}$ of subsets of $X$, called blocks, so that $|A| \in K$ for every $A \in \mathcal{A}$, and every subset $\{x, y\}$ of two points in $X$ is contained in a unique member of $\mathcal{A}$. These may also be called *pairwise balanced designs* (PBDs) with block sizes in $K$.

We use $\alpha(K)$ for the gcd (greatest common divisor) of $\{k-1 : k \in K\}$ and $\beta(K)$ for the gcd of $\{k(k-1) : k \in K\}$. It is known, see [4], that 2-$(n, K, 1)$ designs exist for all integers $n$ that are sufficiently large with respect to $K$ and such that

$$\begin{aligned} n-1 &\equiv 0 \pmod{\alpha(K)}, \\ n(n-1) &\equiv 0 \pmod{\beta(K)}. \end{aligned} \tag{1.1}$$

These congruences are necessary conditions for the existence of a 2-$(n, K, 1)$ design for any $n$.

The following theorem was proved by Colbourn and Rödl in [**2**].

THEOREM 1.1. *Let $K = \{k_1, k_2, \ldots, k_\ell\}$ be given, where the integers $k_i$ are distinct and at least 2. Let $p_1, p_2, \ldots, p_\ell$ be nonnegative real numbers that sum to 1, and let $\epsilon > 0$. For every sufficiently large integer $n$ satisfying* (1.1), *there exists a 2-$(v, K, 1)$ design in which the proportion of blocks of size $k_i$ is within $\epsilon$ of $p_i$, simultaneously for all $i = 1, 2, \ldots, \ell$.*

We give another proof of this theorem. It is no extra work to prove an extension of their result to graph-designs.

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Let $\mathcal{G}$ be a family of simple graphs. A *$\mathcal{G}$-decomposition* of a graph $H$ is a set $\mathcal{D}$ of edge-disjoint subgraphs of $H$, each subgraph in $\mathcal{D}$ isomorphic to a member of $\mathcal{G}$, so that every edge belongs to exactly one member of $\mathcal{D}$. A *$\mathcal{G}$-design* on $n$ points is a $\mathcal{G}$-decomposition of the complete graph $K_n$.

We use $\alpha(\mathcal{G})$ for the gcd of the degrees of the vertices of graphs in $\mathcal{G}$, and $\beta(\mathcal{G})$ for the gcd of $\{2|E(G)| : G \in \mathcal{G}\}$. It is known [3] that $\mathcal{G}$-designs on $n$ points exist for all sufficiently large (with respect to $\mathcal{G}$) integers $n$ satisfying

$$\begin{aligned} n-1 &\equiv 0 \pmod{\alpha(\mathcal{G})}, \\ n(n-1) &\equiv 0 \pmod{\beta(\mathcal{G})}. \end{aligned} \tag{1.2}$$

The congruences (1.2) are necessary conditions for the existence of $\mathcal{G}$-designs on $n$ points for any $n$. In general, if there exists a $\mathcal{G}$-decomposition of $H$, then

$$\begin{aligned} \alpha(\{H\}) &\equiv 0 \pmod{\alpha(\mathcal{G})}, \\ \beta(\{H\}) &\equiv 0 \pmod{\beta(\mathcal{G})}. \end{aligned} \tag{1.3}$$

THEOREM 1.2. *Let $\mathcal{G} = \{G_1, G_2, \ldots, G_\ell\}$ be given, where the graphs $G_i$ are pairwise nonisomorphic and where each has at least one edge. Let $p_1, p_2, \ldots, p_\ell$ be nonnegative real numbers that sum to 1, and let $\epsilon > 0$. For every sufficiently large integer $n$ satisfying the congruences (1.2), there exists a $\mathcal{G}$-design in which the proportion of copies of $G_i$ used in the decomposition is within $\epsilon$ of $p_i$ for all $i = 1, 2, \ldots, \ell$.*

It should be clear that Theorem 1.1 is the consequence of Theorem 1.2, when we take $G_i$ to be a complete graph on $k_i$ points. Theorem 1.2 and a corollary will be proved in Section 2. In Section 3, we prove the following theorem. It is stronger than Theorem 1.2.

THEOREM 1.3. *Let $\mathcal{G} = \{G_1, G_2, \ldots, G_\ell\}$ be given, where the graphs $G_i$ are pairwise nonisomorphic and where each has at least one edge. Let $p_1, p_2, \ldots, p_\ell$ be nonnegative real numbers that sum to 1, and let $\epsilon > 0$. For every sufficiently large integer $n$ satisfying the congruences (1.2), there exists a $\mathcal{G}$-design in which for every point $x$, the proportion of copies of $G_i$ that appear in the decomposition and that contain $x$ is within $\epsilon$ of $p_i$ for all $i = 1, 2, \ldots, \ell$.*

## 2. Proof of Theorem 1.2

Assume the hypothesis and notation of Theorem 1.2. It is sufficient to prove the theorem in the case that the $p_i$'s are rational numbers. Suppose that $p_i = s_i/t$ (with a common denominator $t$) and that $t$ is large enough so that $1/t < \epsilon$.

Let $H_0$ be the vertex-disjoint union of $s_i$ copies of $G_i$, $i = 1, 2 \ldots, \ell$. Let $H_i$ be the vertex-disjoint union of $H_0$ and one additional copy of $G_i$, $i = 1, 2, \ldots, \ell$, and let $\mathcal{H} = \{H_0, H_1, \ldots, H_\ell\}$. In any graph $H_i$, the proportion of copies of $G_i$ that appear is one of $s_i/(t+1)$, $s_i/t$, or $(s_i+1)/(t+1)$. Every $\mathcal{H}$-design immediately gives us a $\mathcal{G}$-design, in which the proportion of copies of $G_i$ that appear is between $s_i/(t+1)$ and $(s_i+1)/(t+1)$, and this is within $\epsilon$ of $p_i$.

From [3], $\mathcal{H}$-designs on $n$ points exist for all large integers $n$ satisfying

$$\begin{aligned} n-1 &\equiv 0 \pmod{\alpha(\mathcal{H})}, \\ n(n-1) &\equiv 0 \pmod{\beta(\mathcal{H})}, \end{aligned} \tag{2.1}$$

We claim that

$$\alpha(\mathcal{H}) = \alpha(\mathcal{G}) \qquad \text{and} \qquad \beta(\mathcal{H}) = \beta(\mathcal{G}), \tag{2.2}$$

i.e. that the congruences (1.2) and (2.1) are identical. This will complete the proof of Theorem 1.2.

The set of degrees of vertices in graphs in $\mathcal{H}$ is identical with the set of degrees of vertices in graphs in $\mathcal{G}$, so the left hand equation in (2.2) is trivial. It is also trivial that $\beta(\mathcal{G})$ divides $\beta(\mathcal{H})$. Finally, since $\beta(\mathcal{H})$ divides $2|E(H_j)|$ for all $j$, it divides the difference

$$2|E(H_i)| - 2|E(H_0)| = 2|E(G_i)|$$

for each $i$. Hence $\beta(\mathcal{H})$ divides the gcd of $2|E(G_i)|$, $i = 1, 2 \ldots, \ell$, which is $\beta(\mathcal{G})$. This establishes the right hand equation in (2.2). □

By a *$G$-packing* $\mathcal{P}$ in $K_n$ we mean a set of edge-disjoint isomorphic copies of $G$ in $K_n$.

COROLLARY 1. *Let $G$ be a simple graph with at least one edge, and let $\epsilon > 0$ be given. For every sufficiently large integer $n$, there exists a $G$-packing $\mathcal{P}$ of $K_n$ so that the ratio of the number edges that occur in copies of $G$ in $\mathcal{P}$ to $n(n-1)/2$ is more than $1 - \epsilon$.*

*Proof.* Apply Theorem 1.2 with $G_1 = G$, with $G_2$ a graph with a single edge, $p_1 = 1$, and $p_2 = 0$. For $\mathcal{G} = \{G_1, G_2\}$, we have $\alpha(\mathcal{G}) = 1$ and $\beta(\mathcal{G}) = 2$, so all integers $n$ satisfy the congruences (1.2) in this case. □

A much stronger result about packings of complete graphs into $K_n$ will appear in [**1**].

## 3. Proof of Theorem 1.3

As in Section 2, we use the fact that if we have an $\mathcal{A}$-decomposition $\mathcal{D}_B$ of each graph $B \in \mathcal{B}$, and a $\mathcal{B}$-decomposition of a graph $H$, then we naturally obtain an $\mathcal{A}$-decomposition $\mathcal{D}$ of $H$, namely

$$\mathcal{D} = \bigcup_{B \in \mathcal{B}} \mathcal{D}_B.$$

Let $A \in \mathcal{A}$ and suppose that for every vertex $x$ of a graph $B \in \mathcal{B}$, the ratio of the number of copies of $A$ in $\mathcal{D}_B$ that contain $x$ to the total number of graph in $\mathcal{D}_B$ that contain $x$ is within $\epsilon$ of a number $p$. Then for every vertex $y$ of $H$, the ratio of the number of copies of $A$ in $\mathcal{D}$ that contain $y$ to the total number of graph in $\mathcal{D}$ that contain $y$ is within $\epsilon$ of $p$.

It is sufficient to prove Theorem 1.3 in the case that the $p_i$'s are rational numbers, and we may also assume that they are positive. Suppose that $p_i = s_i/t$ (with a common denominator $t$, and where $s_i > 0$ for all $i$) and that $t$ is large enough so that $1/t < \epsilon$.

Let $u_i$ be the number of vertices of $G_i$. Let $J$ be the edge disjoint union of $Cs_i/u_i$ copies of $G_i$, $i = 1, 2, \ldots, \ell$, where $C \geq 2$ is an integer chosen so that $Cs_i/u_i$ is an integer for each $i$.

Label the vertices of $J$ with positive integers in the range from 1 to $N$ for some integer $N$ so that the absolute values of the differences of the labels on adjacent vertices are distinct. For example, an (inefficient) way to do this is to use labels $2^0, 2^1, 2^2, \ldots, 2^{v-1}$ in any order, where $v$ is the number of vertices of $J$; here $N =$

$2^{v-1}$. Identify the vertices of $J$ with their labels in the (additive) group $Z_{2N+1}$ of integers modulo $2N+1$, so that $J$ is now a subgraph of the complete graph on vertex set $Z_{2N+1}$.

Let $L_0$ be the union of all translations $J+a$, $a \in Z_{2N+1}$. The condition on absolute values of the differences of labels ensures that the graphs $J+a$ are pairwise edge-disjoint. Because if $x, y \in Z_{2N+1}$ are adjacent in both $J+a$ and $J+b$, then $x-a, y-a$ are adjacent in $J$ and $x-b, y-b$ are adjacent in $J$; if these are not the ends of the same edge of $J$, then $d = \pm(x-y)$ modulo $2N+1$ appears as the difference of the labels of the ends of two edges of $J$; but then $|d|$ is the difference of the labels of the ends of two edges of $J$.

So $L_0$ admits a decomposition into $2N+1$ copies of $J$, and then we obtain a $\mathcal{G}$-decomposition $\mathcal{D}_0$ of $L_0$. The number of translates of a single copy of $G_i$ that contain any given point $x \in Z_{2N+1}$ is $u_i$, so the number of copies of $G_i$ in $\mathcal{D}_0$ that contain any point $x$ is $Cs_i$. The total number of graphs in $\mathcal{D}_0$ that contain $x$ is $Ct$.

Let $L_j$ be obtained from $L_0$ by deleting the edges of one copy of $G_j$ from $L_0$, $i = 1, 2, \ldots, \ell$. Of course, $L_j$ has a $\mathcal{G}$-decomposition $\mathcal{D}_j$ obtained by deleting that one copy of $G_j$ from $\mathcal{D}$. Each point $x \in Z_{2N+1}$ is contained in either $Cs_i$ or $Cs_i - 1$ copies of $G_i$ in $\mathcal{D}_j$; the total number of graphs in $\mathcal{D}_j$ that contain $x$ is $Ct$ or $Ct-1$. In any case, the proportion of copies of $G_i$ among the graphs in $\mathcal{D}_j$ that contain $x$ is between $(Cs_i - 1)/(Ct)$ and $(Cs_i)/(Ct-1)$, and is within $1/t < \epsilon$ of $p_i$.

Let $\mathcal{L} = \{L_0, L_1, \ldots, L_\ell\}$. From [2], $\mathcal{L}$-designs on $n$ points exist for all large integers $n$ satisfying

$$\begin{aligned} n-1 &\equiv 0 \pmod{\alpha(\mathcal{L})}, \\ n(n-1) &\equiv 0 \pmod{\beta(\mathcal{L})},. \end{aligned} \tag{3.1}$$

From a $\mathcal{L}$-design, we obtain a $\mathcal{G}$-design using the $\mathcal{G}$-decompositions of $L_i$ described above. For any point $y$, the proportion of copies of $G_i$ among the graphs in the $\mathcal{G}$-decomposition that contain $y$ will be within $\epsilon$ of $p_i$.

We claim that

$$\alpha(\mathcal{L}) = \alpha(\mathcal{G}) \qquad \text{and} \qquad \beta(\mathcal{L}) = \beta(\mathcal{G}), \tag{3.2}$$

i.e. that the congruences (1.3) and (3.1) are identical. This will complete the proof of Theorem 1.3.

First, since each $L_i$ has a $\mathcal{G}$-decomposition, $\alpha(\mathcal{G})$ divides $\alpha(\{L_i\})$ and $\beta(\mathcal{G})$ divides $\beta(\{L_i\})$ for each $i = 1, 2, \ldots, \ell$. Hence $\alpha(\mathcal{G})$ divides $\alpha(\mathcal{L})$ and $\beta(\mathcal{G})$ divides $\beta(\mathcal{L})$.

If there is a vertex of degree $d$ in some $G_i$, then, since one copy of $G_i$ was deleted from $L_0$ to obtain $L_i$ and $L_0$ is regular of degree $Ct$, then some point in $Z_{2N+1}$ has degree $Ct$ in $L_0$ and degree $Ct - d$ in $L_i$. Then $\alpha(\mathcal{L})$ divides these degrees and so divides the difference $d$. Since this is true for the degree $d$ of every vertex of any graph in $\mathcal{G}$, $\alpha(\mathcal{L})$ divides $\alpha(\mathcal{G})$. Also, $\beta(\mathcal{L})$ divides $2|E(L_0)|$ and $2|E(L_i)|$, so it divides $2|E(L_0)| - 2|E(L_i)| = 2|E(G_i)|$ for each $i$, and hence $\beta(\mathcal{L})$ divides $\beta(\mathcal{G})$. This confirms (3.2). □

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Department of Mathematics, California Institute of Technology, Pasadena, California, 91125

*Current address*: Department of Mathematics, 253-37, California Institute of Technology, Pasadena, California, 91125

*E-mail address*: rmw@caltech.edu